Multiplex sensors and the constant radiance theorem

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Coherent mode representation of the cross-spectral density is used to derive a modal analog of the constant radiance theorem with general applicability to linear optical systems. The theorem is used to consider the relationship between spatial detector geometry and multiplexing capacity.

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Optical sensors, such as cameras and grating spectrometers, are usually designed for isomorphic mappings between physical parameters and measured data. With advances in electronic detector arrays and digital processors, however, sensor systems that are deliberately designed with nonisomorphic mappings are increasingly popular. These are termed "multiplex" systems because they measure linear combinations of target data rather than the data themselves. Multiplexing has a long history in spectroscopy, as in Fourier[1] and Hadamard[2] transform systems, and in x-ray tomography. Motivations for using multiplex spectrometry include the throughput and multiplex advantages. The throughput advantage is that all the power in the target beam is detected and used to generate the target spectrum. The multiplex advantage is that linear combinations of the target data can increase the mean power per measurement and increase the reconstructed signal-to-noise ratio in the presence of additive noise. The multiplex advantage is substantial in the infrared, where thermal noise dominates, but is less compelling in the visible, where shot noise is dominant.[3]

There has recently been considerable interest in multiplex techniques for digital imaging. Conventional multiplexing spectroscopy filters the field through a pinhole, slit, or fiber to reduce the field to a single spatial mode. Spatial filtering may also be used in scanned imaging systems, as in optical coherence tomography.[4] More commonly, however, imaging systems pass multiple spatial modes. We focus our analysis on systems described by coherent mode decompositions of coherence functions. Multiplex imaging has been demonstrated in a variety of multimode systems.[5–10] Multimode multiplexing is used in imaging systems to capture data for which no isomorphic mapping is possible, as in data radiated by three-dimensional sources, or to improve the efficiency of data capture through target-specific mappings of spatial, spectral, polarization, and coherence data. Until recently, the nature of measurable sources and the structure of sensor systems was determined by the nature of analog processing in optical systems. Emerging multiplex systems emphasize efficient data transfer over analog data inversion, under the assumption that inversion can be digitally implemented after data capture.

Some analyses have directly extended the multiplex advantages of spectroscopy to imaging systems. The conventional analysis assumes, however, that detector noise is independent of the multiplexing scheme. A primary goal of this Letter is to show that this assumption cannot hold for multiple-spatial-mode systems. Multiplexing of multiple modes is constrained by the second law of thermodynamics. The second law restricts fan-in and fan-out in optical beams and has broad applications to imaging, solar power collection,[11–13] and optical interconnections.[14]

Second law restrictions on radiance and fan-in transformations of optical beams have been expressed in many forms and with many names. The most common form is the constant radiance theorem.[15] These theorems are most commonly derived by use of ray optics[12] but have also been derived by use of wave theory[16] and thermodynamic arguments.[11] As suggested by the name, the constant radiance theorem shows that no linear optical system can increase the radiance in transformations between incoherent planes.

In isomorphic imaging systems, there is a direct analogy between the physical state of the field at significant points and source parameters, and one can easily prove the constant radiance theorem by ray arguments. The situation is more complex in multiplex imaging because there need not be any particular physical significance to the field at any point. In this Letter an alternative version of second law constraints is developed that allows analysis of systems based on any linear optical transformations and in arbitrary coherence states. This goal is achieved by use of the modal theory of partial coherence developed by Wolf.[17]

The discussion is limited to fields propagating in source-free unbounded homogeneous media, in which case second-order coherence functions of the radiant field are completely determined by measures between points on a bounding surface. We can describe the states of these fields and transformations of them by use of the cross-spectral density, $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$, and the appropriate derivatives for points $\mathbf{r}_1$ and $\mathbf{r}_2$ across a surface bounding the source.

The cross-spectral density is defined as the Fourier transform of the mutual coherence function, $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$.[18] Since $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is Hermitian and positive definite in transformations of functions of

$\mathbf{r}_1$ and $\mathbf{r}_2$, it can be represented by a coherent mode expansion of the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_n \lambda_n(\nu)\phi_n{}^*(\mathbf{r}_1, \nu)\phi_n(\mathbf{r}_2, \nu)\,, \tag{1}$$

where $\lambda_n(\nu)$ is real and positive and where the family of functions $\phi_n(\mathbf{r}, \nu)$ are orthonormal such that $\int \phi_m{}^*(\mathbf{r}, \nu)\phi_n(\mathbf{r}, \nu)\mathrm{d}^2r = \delta_{mn}$.

In analogy with radiance transformations, we are interested in transformations of the coherent mode distribution for fields propagating between planes. A linear optical system transforms coherent modes defined on an input surface into distributions on an output surface under the impulse response $h(\mathbf{r}, \mathbf{r}', \nu)$, where $\mathbf{r}$ and $\mathbf{r}'$ are input and output position vectors, respectively. The cross-spectral density is transformed by propagation through the system into

$$W(\mathbf{r}_1{}', \mathbf{r}_2{}', \nu) = \sum_n \lambda_n(\nu)\psi_n{}^*(\mathbf{r}_1{}', \nu)\psi_n(\mathbf{r}_2{}', \nu)\,, \tag{2}$$

where $\psi_n(\mathbf{r}', \nu) = \int \phi_n(\mathbf{r}, \nu)h(\mathbf{r}, \mathbf{r}', \nu)\mathrm{d}^2r$. $\mathbf{r}_1{}'$ and $\mathbf{r}_2{}'$ correspond to points on the output surface of the system. The functions $\psi_n(\mathbf{r}, \nu)$ are not necessarily orthogonal.[19] The cross-spectral density across the output aperture may be described by a new coherent mode decomposition:

$$W(\mathbf{r}_1{}', \mathbf{r}_2{}', \nu) = \sum_n \Lambda_n(\nu)\Phi_n{}^*(\mathbf{r}_1{}', \nu)\Phi_n(\mathbf{r}_2{}', \nu)\,, \tag{3}$$

where the functions $\Phi_n(\mathbf{r}, \nu)$ are a new set of orthonormal coherent modes and the functions $\Lambda_n(\nu)$ are new eigenvalues. The new coherent modes are complete over the possible states of the field in the output plane, meaning that the states $\psi_n(\mathbf{r}, \nu)$ can be expanded as $\psi_n(\mathbf{r}, \nu) = \sum_m c_{nm}\Phi_m(\mathbf{r}, \nu)$. Using the orthonormality of the coherent modes, we know from Eq. (3) that

$$\int W(\mathbf{r}_1{}', \mathbf{r}_2{}', \nu)\Phi_m(\mathbf{r}_1{}', \nu)\Phi_m{}^*(\mathbf{r}_2{}', \nu)\mathrm{d}^2r_1{}'\mathrm{d}r_2{}' = \Lambda_m(\nu)\,. \tag{4}$$

Substituting W from Eq. (2) into Eq. (4), we find that

$$\Lambda_m(\nu) = \sum_n |c_{nm}|^2\lambda_n\,. \tag{5}$$

Power conservation requirements allow us to constrain the transformation coefficients, c_{nm}. Conservation of power on propagation through the system leads to the requirement that $\int \psi_n{}^*(\mathbf{r}, \nu)\psi_n(\mathbf{r}, \nu)\mathrm{d}^2r \le 1$, which implies that $\sum_m |c_{nm}|^2 \le 1$.

$|c_{nm}|^2$ has the properties of a probability distribution. From the weighting that is implicit in this distribution, we can see immediately that no linear transformation can increase the maximum mode amplitude, which is to say that $[\Lambda_m]_{\max} \le [\lambda_n]_{\max}$. This result is analogous to the constant radiance theorem in that the brightest possible focused spot drawn from the field will be proportional to amplitude of the brightest mode.

If we have no prior knowledge of the original eigenvalues, as in cases for which the values λ_n represent encodable communications, image, or memory data, the distributions of λ_n can be taken as a measure of the entropy of the system. Defining the normalized eigenvalue $\tilde{\lambda}_m = \lambda_m/\sum_n \lambda_n$, we construct the entropy of the system as[20]

$$H = -\sum_n \tilde{\lambda}_m \log \tilde{\lambda}_m. \tag{6}$$

One can show by induction that any change in $\tilde{\lambda}_m$ that redistributes the largest $\tilde{\lambda}_m$ into smaller ranges will increase H. Thus, the proof that one cannot increase the largest eigenvalue is simply an expression of the second law of thermodynamics. Of course, linear optical transformations are generally reversible and thus ought not to transform the entropy at all. Any transformation that changes the eigenvalue spectrum must be irreversible to satisfy the second law. In practical systems, irreversibility arises from many factors, including phase loss and nonlinearity on absorption from the field, segmentation of field regions by hard obscurations, and slight mechanical instabilities.

In application of these results to multiplex imaging, a spatially parallel array of square-law spatially and temporally integrating detectors, is considered. The state of the ith such detector can be modeled as

$$m_i = \int \mathrm{d}\nu \int_{A_i} S(\mathbf{r}_s, \nu)\kappa(\nu)\mathrm{d}^2r_s = \sum\nolimits_j \int \beta_{ij}(\nu)\lambda_j(\nu)\mathrm{d}\nu\,, \tag{7}$$

where the spatial integral is over the detector area, A_i, and the spectral integral is over the entire spectrum. $\kappa(\nu)$ is the spectral efficiency of the detector. $\mathbf{r}_s$ is a position vector on the detector surface. $S(\mathbf{r}_s, \nu) = W(\mathbf{r}_s, \mathbf{r}_s, \nu)$ is the power spectral density evaluated at $\mathbf{r}_s$. $W(\mathbf{r}_s, \mathbf{r}_s, \nu)$ has been transformed on propagation, as discussed above. The power coupling coefficient from the jth coherent mode in the detector plane to the ith detector is $\beta_{ij}(\nu) = \kappa(\nu)\int_{A_i} |\phi_j(\mathbf{r}_s, \nu)|^2\mathrm{d}^2r_s$. Using the orthonormality of the coherent modes, as we know that $\sum_{ij} \beta_{ij}(\nu) \le \kappa(\nu)N$, where N is the number of modes and equality applies if and only if the aggregate detector integration area covers the entire sensor plane. If we assume that the modal distribution is uniform over the sensor plane, then we can segment the coupling coefficients to obtain

$$\sum_j \beta_{ij}(\nu) \le \frac{\kappa(\nu)NA}{A_s}\,, \tag{8}$$

where A_s is the total area of the sensor plane.

In conventional multiplex spectroscopy, the input field is single mode and measurements are of the form $m_i = \int \beta_i(\nu)\lambda(\nu)\mathrm{d}\nu$. Only power conservation constraints apply to the coupling coefficient in these systems, i.e., $\beta_i(\nu) \le \kappa_{\max}$. Multiplex imaging is complicated by the following factors.

- Source data are encoded both in the mode coefficients, $\lambda_i(\nu)$, and in the coupling coefficients, $\beta_{ij}(\nu)$.

In conventional systems, $\beta(\nu)$ is independent of the source state. The relationship between the modes and the coupling coefficients in multiplex systems has the effect of making sensing more difficult and, through the combination of the restriction that $\sum_m |c_{nm}|^2 \le 1$ and Eq. (8), limiting the power on individual sensor elements.

- The range of $\beta_{ij}(\nu)$ is constrained by Eq. (8). In conventional multiplexing, $\beta(\nu)$ is not strongly coupled to detector size or geometry. As expressed in in Eq. (8), multiplexing data from more than one mode is power efficient only if the detector size grows with the number of modes that are multiplexed. Since detector noise and bandwidth are not independent of detector size, conventional analyses of the multiplex advantage do not necessarily apply to multiplex imaging systems.

Multiplex systems may be subdivided into single-mode spectrometers, systems such as planar hyperspectral imagers in which the coherent modes are known, and systems such as multidimensional spatiospectral sensors in which the coherent modes are unknown. The primary advantage of multiplexing in single-mode systems is that the net detected power is greater under multiplexing, thereby improving photon efficiency and signal-to-noise ratio. The constant radiance theorem suggests that conventional advantages of multiplex sensing are less persuasive in the plane-to-plane imaging case because multiplexing one cannot necessarily increase the power on individual detector elements without increasing the detector area. Depending on detector size and source statistics, however, one may still achieve a multiplexing advantage in planar imaging systems. Comparison of potential advantages with alternative schemes, such as adaptive sensors, is a challenge for future research. Adaptive filtering to discover and match the source modes is the basis of adaptive optical telescopy. Adaptive optical systems correct for relatively weak global distortions of the coherent mode structure. Adaptive filtering for strong distortions and general partially coherent fields has not yet been demonstrated. One might also consider a strategy of combining mode powers on absorption through fluorescent mode reduction strategies to address the constant radiance theorem. Entropy constraints in this case will be similar to those in solar power collection.[13]

When either the coherent modes are unknown or one chooses not to filter them, $\lambda_1(\nu)$ and $\beta_{ij}(\nu)$ are unknown. The number of measurements required for resolution of these variables depends on the extent of prior knowledge. In most cases, one chooses to vary the optical system to generate a full rank linear relationship between the measurements and the source field. Examples of systems that can measure a full rank transformation include direct[9] and indirect[10] coherence measurement systems. How well conditioned this relationship is depends both on the nature of the source and on the set of optical transformations implemented. The selection of optical transformations or implementation of adaptive systems to achieve well-conditioned transformations is the key challenge in the future design of multiplex imagers.

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