The inverse problem for random sources[a)]

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The problem of deducing the statistical structure of a localized random source $\rho(\mathbf{r})$ of the reduced wave equation from measurements of the field external to the source is addressed for the case when the measurements yield the autocorrelation function of the field at all pairs of points exterior to the source volume and the quantity to be determined is the source's autocorrelation function $R_\rho(\mathbf{r}_1,\mathbf{r}_2) = \langle\rho^*(\mathbf{r}_1)\rho(\mathbf{r}_2)\rangle$. This problem is shown to be equivalent to that of determining R_ρ from the autocorrelation function of the field's radiation pattern and is found, in general, not to admit a unique solution due to the possible existence of nonradiating sources within the source volume. Notable exceptions are the class of delta correlated (incoherent) sources whose intensity profiles are shown to be uniquely determined from the data and the class of quasihomogeneous sources whose coherence properties can be determined if their intensity profiles are known and vice versa.

1. INTRODUCTION

An inverse problem of interest in optics and acoustics is that of deducing a deterministic source $\rho(\mathbf{r})$ of the reduced wave equation

$$(\nabla^2 + k_0^2)\psi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \tag{1.1}$$

from measurements of the field ψ at points external to the region of localization V of the source. In the most favorable case ψ will be exactly known everywhere outside V. In this case the inverse problem reduces to that of determining $\rho(\mathbf{r})$ from the value of its Fourier transform

$$\tilde{\rho}(\mathbf{k}) = \int_V d^3r\,\rho(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{1.2}$$

evaluated in $\mathbf{k}$ space on the surface of a sphere centered at the origin ($\mathbf{k} = \mathbf{0}$) and of radius k_0. This conclusion follows from the fact that $\psi(\mathbf{r})$ is uniquely determined everywhere outside V by its *radiation pattern*,[1] which, in turn, is equal[2] to the above stated boundary value of the sources transform; i.e.,

$$\psi(\mathbf{r}) \sim \tilde{\rho}(k_0\hat{\mathbf{r}})e^{ik_0r}/r, \quad (k_0r \to \infty), \tag{1.3}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$.

The inverse problem described above does not admit a unique solution due to the possible existence of so-called *nonradiating sources*[3] within the source volume. Such sources possess Fourier transforms which vanish identically when $|\mathbf{k}| = k_0$ and thus produce fields which vanish everywhere outside their region of localization. It follows that solutions to the inverse source problem can be determined only up to an unknown additive nonradiating part which must be specified by information other than field data.

In many applications the source $\rho(\mathbf{r})$ and, hence, the field $\psi(\mathbf{r})$ will not be deterministic (i.e., perfectly coherent) but rather will be realizations of random processes that are characterized by the source autocorrelation function[4]

$$R_\rho(\mathbf{r}_1,\mathbf{r}_2) = \langle\rho^*(\mathbf{r}_1)\rho(\mathbf{r}_2)\rangle \tag{1.4}$$

and higher order moments. Within this context of "random sources" the inverse problem becomes that of deducing the *statistical structure* of the source from physically realizable measurements of the radiated field.

In this paper we address the inverse problem for random sources for the case when the measurements yield the autocorrelation function[5] of the field

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2) = \langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle \tag{1.5}$$

at all pairs of points exterior to the source volume and the quantity to be determined is the source's autocorrelation function $R_\rho(\mathbf{r}_1,\mathbf{r}_2)$. In Sec. 2 it is shown that the autocorrelation function of the field is uniquely determined everywhere outside the source region by the autocorrelation function of the radiation pattern and vice-versa. This latter quantity is shown to be equal to the (six-dimensional) Fourier spectrum of the source autocorrelation function[6]

$$\Phi(\mathbf{k}_1,\mathbf{k}_2) = \langle\tilde{\rho}^*(\mathbf{k}_1)\tilde{\rho}(\mathbf{k}_2)\rangle \tag{1.6}$$

evaluated on the four-dimensional surface $|\mathbf{k}_1| = |\mathbf{k}_2| = k_0$. The inverse problem for random sources is thus found to reduce to that of determining $\Phi(\mathbf{k}_1,\mathbf{k}_2)$ [and, hence, $R_\rho(\mathbf{r}_1,\mathbf{r}_2)$] from its boundary value on the surface $|\mathbf{k}_1| = |\mathbf{k}_2| = k_0$.

It is shown in Sec. 3 that it is not possible in general to uniquely determine the spectrum $\phi(\mathbf{k}_1,\mathbf{k}_2)$ from its boundary value as determined by the autocorrelation function of the radiation pattern. Thus, like its deterministic counterpart, the inverse problem for random sources does not in general admit a unique solution. A notable exception is found to be the class of delta correlated sources for which the intensity profile $\langle|\rho(\mathbf{r})|^2\rangle$ can be uniquely determined from the known boundary value of $\Phi(\mathbf{k}_1,\mathbf{k}_2)$.

Finally, in Sec. 4, the inverse problem is formulated for so-called quasihomogeneous sources.[7] These sources are locally statistically homogeneous and are characterized by an autocorrelation function which factors into the product of the intensity profile of the source with a normalized autocor-

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relation function $g(\mathbf{r}_2 - \mathbf{r}_1)$ of a strictly statistically homogeneous source. It is shown for such sources that if either the intensity profile or the normalized autocorrelation function $g(\mathbf{r}_2 - \mathbf{r}_1)$ is known, the other can be uniquely determined from the autocorrelation function of the radiation pattern.

2. FORMULATION OF THE INVERSE PROBLEM

Throughout this paper we shall restrict our attention to that class of random sources whose realizations $\rho(\mathbf{r})$ are piecewise continuous and localized within a volume V which, for convenience, we take to be a sphere of radius R_0 centered at the origin. The field generated by any such realization is identified with that particular solution of Eq. (1.1) which behaves as an outgoing spherical wave at infinity, i.e., such that

$$\psi(\mathbf{r}) \sim \hat{\psi}(\hat{\mathbf{r}}) e^{ik_0 r}/r, \quad (k_0 r \to \infty), \tag{2.1}$$

where $\hat{\psi}(\hat{\mathbf{r}})$ is the radiation pattern of the field evaluated in the direction $\hat{\mathbf{r}}$ of the field point $\mathbf{r} = r\hat{\mathbf{r}}$. The appropriate solution of Eq. (1.1) satisfying the asymptotic condition (2.1) is

$$\psi(\mathbf{r}) = \int_V d^3r' \, \rho(\mathbf{r}') e^{ik_0|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|. \tag{2.2}$$

It is easily verified that the field given in Eq. (2.2) satisfies the asymptotic condition (2.1) with the radiation pattern given by

$$\hat{\psi}(\hat{\mathbf{r}}) = \int_V d^3r' \, \rho(\mathbf{r}') e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'} = \tilde{\rho}(k_0\hat{\mathbf{r}}). \tag{2.3}$$

As mentioned in the Introduction the radiation pattern $\hat{\psi}(\hat{\mathbf{r}})$ not only yields information about the far field but, in fact, uniquely specifies $\psi(\mathbf{r})$ everywhere outside the source volume and vice versa.[2] This one-to-one correspondence between the radiation pattern and the value of the field outside V is extremely important for the inverse problem and, thus, will now be established with the aid of the well-known multipole expansion of the field. This expansion, which can be obtained by expanding the Green function appearing in Eq. (2.2) into a series of spherical wave eigenfunctions of the reduced wave equation, converges everywhere outside V (i.e., for $r > R_0$) and is given by[8]

$$\psi(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n(k_0 r) Y_n^m(\theta,\phi). \tag{2.4}$$

Here $h_n(k_0 r)$ is the spherical Hankel function of the first kind of order n, $Y_n^m(\theta,\phi)$ is the spherical harmonic of degree n and order m, and (r,θ,ϕ) are the spherical polar coordinates of the field point $\mathbf{r}$. The expansion coefficients (multipole moments) a_n^m can be determined from the value of the field given over the surface of a sphere of radius $R > R_0$ by means of the formula

$$a_n^m = \frac{1}{h_n(k_0 R)} \int_{-\pi}^{\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \psi(R\hat{\mathbf{r}}) Y_n^{m^*}(\theta,\phi). \tag{2.5}$$

It follows from Eqs. (2.4) and (2.5) that there is a one-to-one correspondence between the set of multipole moments $\{a_n^m\}$ $(n = 0, 1,...; m = -n, -n+1,...,n)$ and the value of the field specified at all points lying outside the source volume. Moreover, by asymptotically expanding both sides of Eq. (2.4) we find that

$$\hat{\psi}(\hat{\mathbf{r}}) = \frac{1}{k_0} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m(\theta,\phi), \tag{2.6}$$

from which we conclude that there is also a one-to-one correspondence between the set of multipole moments and the radiation pattern and, hence, between the radiation pattern and the value of the field at all points lying outside V.

The deterministic inverse source problem is that of deducing the source $\rho(\mathbf{r}')$ from the field $\psi(\mathbf{r})$ specified everywhere outside the source volume. This problem thus consists mathematically of solving the integral equation (2.2) for $\rho(\mathbf{r}')$ in terms of the value of $\psi(\mathbf{r})$ specified everywhere outside V. However, the one-to-one correspondence between the value of the field outside V and the radiation pattern shows that this problem is equivalent to that of determining the source transform $\tilde{\rho}(\mathbf{k})$ for all values of the wave vector $\mathbf{k}$ from its boundary value $\tilde{\rho}(k_0\hat{\mathbf{r}})$ as given by the radiation pattern via Eq. (2.3).

When the source $\rho(\mathbf{r})$ is a random process the inverse problem becomes that of deducing the statistical structure of this process from physically realizable field measurements performed exterior to the source region. In practice, especially at optical frequencies, the field measurements will consist of interference experiments from which the autocorrelation function $\Gamma(\mathbf{r}_1,\mathbf{r}_2)$ [cf. Eq. (1.5)] can be determined. In the ideal case, which we address here, $\Gamma(\mathbf{r}_1,\mathbf{r}_2)$ will be known for all pairs of points lying outside the source volume. It is natural then to define the inverse problem for random sources to be that of deducing the source autocorrelation from this information. Mathematically, this consists of solving the integral equation

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2) = \int_V d^3r_1' \int_V d^3r_2' \, R_\rho(\mathbf{r}_1',\mathbf{r}_2') \times \frac{e^{ik_0|\mathbf{r}_2-\mathbf{r}_2'|}}{|\mathbf{r}_2-\mathbf{r}_2'|} \cdot \frac{e^{-ik_0|\mathbf{r}_1-\mathbf{r}_1'|}}{|\mathbf{r}_1-\mathbf{r}_1'|} \tag{2.7}$$

for $R_\rho(\mathbf{r}_1',\mathbf{r}_2')$ in terms of $\Gamma(\mathbf{r}_1,\mathbf{r}_2)$ given for all pairs of points outside V, and thus is completely analogous to the deterministic problem of inverting Eq. (2.2) for $\rho(\mathbf{r}')$ given $\psi(\mathbf{r})$ everywhere outside V.

It follows from Eqs. (2.4) and (2.5) that the autocorrelation function of the field at all pairs of points external to the source region is uniquely specified by the various statistical moments $\langle a_n^{m*} a_{n'}^{m'} \rangle$ and vice versa. In virtue of Eq. (2.6) a similar one-to-one correspondence exists between these statistical moments and the autocorrelation function $\langle \hat{\psi}^*(\hat{\mathbf{r}}_1) \hat{\psi}(\hat{\mathbf{r}}_2) \rangle$ of the radiation pattern. We conclude then that *the autocorrelation function of the field is uniquely determined everywhere outside the source volume by the autocorrelation function of the radiation pattern and vice versa.* It follows that the inverse problem for random sources reduces, in analogy to its deterministic counterpart, to that of deducing the autocorrelation function of the source from the autocorrelation function of the radiation pattern.

A more complete analogy between the deterministic and random inverse source problems is obtained by introducing the sixfold Fourier transform of the source autocorrelation function

$$\Phi(\mathbf{k}_1,\mathbf{k}_2)=\int d^3r_1\int d^3r_2\, R_\rho(\mathbf{r}_1,\mathbf{r}_2)e^{-i(\mathbf{k}_2\cdot\mathbf{r}_2-\mathbf{k}_1\cdot\mathbf{r}_1)} \quad (2.8)$$

This quantity is readily shown to be equal to the autocorrelation function of the Fourier transform of the source

$$\Phi(\mathbf{k}_1,\mathbf{k}_2)=\langle\tilde{\rho}^*(\mathbf{k}_1)\tilde{\rho}(\mathbf{k}_2)\rangle. \quad (2.9)$$

The inverse problem for random sources thus becomes that of deducing $\Phi(\mathbf{k}_1,\mathbf{k}_2)$ for all values of the wave vectors $\mathbf{k}_1,\mathbf{k}_2$ from its boundary value

$$\Phi(k_0\hat{\mathbf{r}}_1,k_0\hat{\mathbf{r}}_2)=\langle\hat{\psi}^*(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)\rangle. \quad (2.10)$$

as given by the autocorrelation function of the radiation pattern.

3. NONUNIQUENESS IN THE INVERSE PROBLEM

In both the deterministic and random inverse source problems the data has lower *dimensionality* than what is required to uniquely specify the source in question. For example, in the deterministic case the data consists of the prescription of an unknown function $\tilde{\rho}(\mathbf{k})$ of *three* variables (say the Cartesian components of $\mathbf{k}$) on the *two-dimensional* surface $|\mathbf{k}|=k_0$. It is not possible to uniquely continue $\tilde{\rho}(\mathbf{k})$ from its boundary value over such a two-dimensional surface even for the class of sources considered here (localized and continuous) whose transforms are entire analytic functions of $\mathbf{k}$.[9] In particular, the entire function $\tilde{\rho}(\mathbf{k})$ is uniquely specified for all values of $\mathbf{k}$ if and only if it is specified over a finite *volume element* in $\mathbf{k}$ space,[10] i.e., over a three-dimensional region. The data in the deterministic inverse source problem specifies $\tilde{\rho}(\mathbf{k})$ only over a two-dimensional surface and, thus, is not sufficient to uniquely determine this quantity for all values of $\mathbf{k}$.

A similar situation prevails in the inverse problem for random sources. In this case the data is a prescription of the entire analytic function $\Phi(\mathbf{k}_1,\mathbf{k}_2)$ of *six* variables over the four dimensional surface $|\mathbf{k}_1|=|\mathbf{k}_2|=k_0$ while unique determination requires that it be specified over a finite volume element in $(\mathbf{k}_1,\mathbf{k}_2)$ space, i.e., over a *six-dimensional* region.

Examples of deterministic sources that are not uniquely specified from their values on the surface $|\mathbf{k}|=k_0$ are provided by the class of so-called *nonradiating sources*.[3] These sources are readily constructed by applying the operator $(\nabla^2+k_0^2)$ to any thrice differentiable function $Q(\mathbf{r})$ localized within V. In particular the source

$$\rho_{\text{N.R.}}(\mathbf{r})=(\nabla^2+k_0^2)Q(\mathbf{r}) \quad (3.1)$$

possesses the Fourier transform

$$\tilde{\rho}_{\text{N.R.}}(\mathbf{k})=\int d^3r[(\nabla^2+k_0^2)Q(\mathbf{r})]e^{-i\mathbf{k}\cdot\mathbf{r}} = -(k^2-k_0^2)\tilde{Q}(\mathbf{k}) \quad (3.2)$$

with $\tilde{Q}(\mathbf{k})$ being the Fourier transform of $Q(\mathbf{r})$. It is seen from Eq. (3.2) that irrespective of the choice of the function $Q(\mathbf{r})$ the source transform $\tilde{\rho}_{\text{N.R.}}(\mathbf{k})$ vanishes when $|\mathbf{k}|=k_0$. The class of nonradiating sources thus possess transforms which are identical on the surface $|\mathbf{k}|=k_0$ (namely, zero) but are rather arbitrary for values of $\mathbf{k}$ not on this surface.

The fact that a nonradiating source $\rho_{\text{N.R.}}(\mathbf{r})$ possesses a transform which vanishes when $|\mathbf{k}|=k_0$ means that it generates a field which vanishes identically outside its region of localization. It follows that the deterministic sources $\rho(\mathbf{r})$ and $\rho(\mathbf{r})+\rho_{\text{N.R.}}(\mathbf{r})$ produce identical fields outside V and, hence, are both solutions to the same inverse source problem. The lack of uniqueness of solutions to the deterministic inverse source problem can thus be viewed as being due to the possible existence of nonradiating sources within the source volume.[11]

The lack of uniqueness of solutions to the random inverse source problem can likewise be be attributed to the possible existence of nonradiating sources within the source volume. In this case the function $Q(\mathbf{r})$ appearing in Eq. (3.1) is chosen to be a member of a random ensemble of thrice differentiable functions localized within V. The six-dimensional Fourier spectrum of the autocorrelation function of the random source

$$\rho'(\mathbf{r})=\rho(\mathbf{r})+\rho_{\text{N.R.}}(\mathbf{r}) \quad (3.3)$$

is found to be

$$\begin{aligned}\Phi'(\mathbf{k}_1,\mathbf{k}_2)&=\langle\tilde{\rho}^*(\mathbf{k}_1)\tilde{\rho}(\mathbf{k}_2)\rangle+\langle\tilde{\rho}^*(\mathbf{k}_1)\tilde{\rho}_{\text{N.R.}}(\mathbf{k}_2)\rangle\\&\quad+\langle\tilde{\rho}^*_{\text{N.R.}}(\mathbf{k}_1)\tilde{\rho}(\mathbf{k}_2)\rangle+\langle\tilde{\rho}^*_{\text{N.R.}}(\mathbf{k}_1)\tilde{\rho}_{\text{N.R.}}(\mathbf{k}_2)\rangle\\&=\Phi(\mathbf{k}_1,\mathbf{k}_2)-(k_2^2-k_0^2)\langle\tilde{\rho}^*(\mathbf{k}_1)\tilde{Q}(\mathbf{k}_2)\rangle\\&\quad-(k_1^2-k_0^2)\langle\tilde{Q}^*(\mathbf{k}_1)\tilde{\rho}(\mathbf{k}_2)\rangle\\&\quad+(k_1^2-k_0^2)(k_2^2-k_0^2)\langle\tilde{Q}^*(\mathbf{k}_1)\tilde{Q}(\mathbf{k}_2)\rangle. \quad (3.4)\end{aligned}$$

We conclude that

$$\Phi'(\mathbf{k}_1,\mathbf{k}_2)\big|_{|\mathbf{k}_1|=|\mathbf{k}_2|=k_0}=\Phi(\mathbf{k}_1,\mathbf{k}_2)\big|_{|\mathbf{k}_1|=|\mathbf{k}_2|=k_0} \quad (3.5)$$

independent of the statistical structure of the random process $Q(\mathbf{r})$. Thus, the two random sources $\rho(\mathbf{r})$ and $\rho'(\mathbf{r})$ generate fields possessing identical autocorrelation functions outside V and, thus, are both solutions to the same inverse problem.[12]

The discussion presented above assumes, of course, that the only information available concerning the unknown source is that generated by field measurements performed *exterior* to the source volume. If field measurements are allowed *internal* to the source region or if additional information is available which reduces the dimensionality of the source, then the inverse problem may possess a unique solution.

An example of the latter situation is provided by the class of *incoherent sources*. These sources possess autocorrelation functions of the form

$$R_\rho(\mathbf{r}_1,\mathbf{r}_2)=I(\mathbf{r}_1)\delta(\mathbf{r}_2-\mathbf{r}_1), \quad (3.6)$$

where $I(\mathbf{r}_1)$ is a continuous, nonnegative function called the "intensity profile" of the source and $\delta(\mathbf{r}_2-\mathbf{r}_1)$ is the three-dimensional Dirac delta function. The Fourier transform of $R_\rho(\mathbf{r}_1,\mathbf{r}_2)$ is found to be

$$\Phi(\mathbf{k}_1,\mathbf{k}_2)=\int_V d^3r_1\, I(\mathbf{r}_1)e^{-i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}_1}=\tilde{I}(\mathbf{k}_2-\mathbf{k}_1). \quad (3.7)$$

It is seen from Eq. (3.7) that an incoherent source is characterized by an entire analytic function $\tilde{I}(\mathbf{k})$ of only *three variables*. Moreover,

$$\Phi(k_0\hat{\mathbf{r}}_1,k_0\hat{\mathbf{r}}_2)=\tilde{I}[k_0(\hat{\mathbf{r}}_2-\hat{\mathbf{r}}_1)], \quad (3.8)$$

which shows that $\tilde{I}(\mathbf{k})$ is specified for all values of $\mathbf{k}$ lying within a sphere of radius $2k_0$ by the autocorrelation function of the radiation pattern. It follows that $\tilde{I}(\mathbf{k})$ is, in principle, completely determined (e.g., by analytic continuation) so that the inverse problem for incoherent sources admits a unique solution.

4. QUASIHOMOGENEOUS SOURCES

The concept of a quasihomogeneous source[7] is a natural generalization of the concept of statistical homogeneity to include sources of finite extent. The term "statistically homogeneous" is, of course, used to denote multidimensional, *stationary* random processes which, by definition, cannot be localized to a finite region. Carter and Wolf [7] argued, however, that many physical sources behave *locally* as though they were statistically homogeneous and thus have autocorrelation functions that can be approximated as follows:

$$R_\rho(\mathbf{r}_1,\mathbf{r}_2)=I[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)]g(\mathbf{r}_2-\mathbf{r}_1). \quad (4.1)$$

The function $g(\mathbf{r})$ is a measure of the spatial coherence of the source and is assumed to be appreciably different from zero only for values of its argument $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ lying in some domain D which is much smaller than the source volume V. The quantity $I(\mathbf{R})$, called the source intensity, is assumed to be nonnegative, is a slowly varying function of its argument $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$ over the source volume V and vanishes outside this region. In addition, $I(\mathbf{R})$ is assumed to be essentially constant over the volume of coherence D. Sources having autocorrelation functions that can be approximated by the above model are called *quasihomogeneous.*

Substituting Eq. (4.1) into Eq. (2.8) yields the following expression for the sixfold Fourier transform of the autocorrelation function of a quasihomogeneous source:

$$\Phi(\mathbf{k}_1,\mathbf{k}_2)=\int d^3r_1\int d^3r_2\, I[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)] \times g(\mathbf{r}_2-\mathbf{r}_1)e^{-i(\mathbf{k}_2\cdot\mathbf{r}_2-\mathbf{k}_1\cdot\mathbf{r}_1)}. \quad (4.2)$$

The above expression is simplified considerably by changing the variables of integration from $(\mathbf{r}_1,\mathbf{r}_2)$ to the $(\mathbf{R},\mathbf{r})$ variables defined above. Performing this change of variables yields[13]

$$\Phi(\mathbf{k}_1,\mathbf{k}_2)=\int d^3R\int d^3r\, I(\mathbf{R})g(\mathbf{r}) \times e^{-i\left\{(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{R}+\left[\frac{\mathbf{k}_1+\mathbf{k}_2}{2}\right]\cdot\mathbf{r}\right\}} =\tilde{I}(\mathbf{k})\tilde{g}(\mathbf{K}), \quad (4.3)$$

where we have introduced the wave vectors

$$\mathbf{k}=\mathbf{k}_2-\mathbf{k}_1,\quad \mathbf{K}=\tfrac{1}{2}(\mathbf{k}_1+\mathbf{k}_2) \quad (4.4)$$

and where $\tilde{I}(\mathbf{k})$ and $\tilde{g}(\mathbf{K})$ are the Fourier transforms of $I(\mathbf{R})$ and $g(\mathbf{r})$ respectively.

The data for the inverse problem specifies $\Phi(\mathbf{k}_1,\mathbf{k}_2)$ on the boundary $|\mathbf{k}_1|=|\mathbf{k}_2|=k_0$. It follows from Eqs. (4.4) that this surface corresponds to the boundary

$$k^2+4K^2=4k_0^2 \quad (4.5)$$

in $(\mathbf{k},\mathbf{K})$ space. The inverse problem for quasihomogeneous sources can thus be stated as follows: *determine an intensity profile* $I(\mathbf{R})$ *and coherence function* $g(\mathbf{r})$ *having Fourier transforms* $\tilde{I}(\mathbf{k})$ *and* $\tilde{g}(\mathbf{K})$, *respectively, the product of which assumes a specified value (equal to the autocorrrelation function of the radiation pattern) for all values of* $(\mathbf{k},\mathbf{K})$ *lying on the boundary defined in Eq.* (4.5).

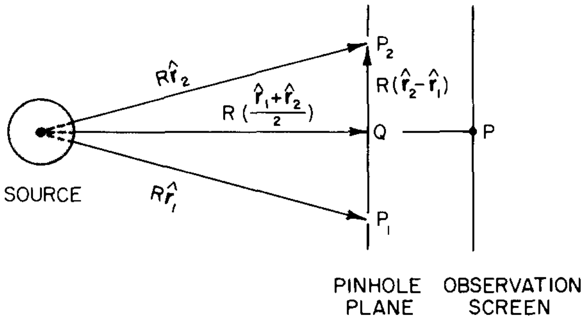

FIG. 1. Young's interference experiment. The autocorrelation function of the radiation pattern in the directions $\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2$ is determined by measuring the fringes in the vicinity of the point P on the observation screen.

The uniqueness question in the inverse problem for quasihomogeneous sources is complicated considerably by the requirements that the spectrum $\Phi(\mathbf{k}_1,\mathbf{k}_2)$ be factorizable in the form required by Eq. (4.3) and that the transforms $I(\mathbf{R})$ and $g(\mathbf{r})$ of the two factors $\tilde{I}(\mathbf{k})$ and $\tilde{g}(\mathbf{K})$, respectively, possess the properties described earlier. If, however, we restrict our attention to sources for which the two functions $\tilde{I}(\mathbf{k})$ and $\tilde{g}(\mathbf{K})$ are analytic and for which one of these two functions is known, then it is not difficult to show that the inverse problem admits a unique solution. For example, if $\tilde{I}(\mathbf{k})$ is known, then we conclude from the statement of the inverse problem given above that $\tilde{g}(\mathbf{K})$ can be determined over a *volume* element in $\mathbf{K}$ space, namely for all values of $\mathbf{K}$ lying within the sphere, centered at $\mathbf{K}=\mathbf{0}$, and of radius k_0. Analyticity of $\tilde{g}(\mathbf{K})$ then allows this quantity to be determined everywhere by analytic continuation.

The situation considered above where either the intensity profile $I(\mathbf{R})$ or coherence function $g(\mathbf{r})$ is known is of great interest both because it occurs frequently in practice and because for such cases the unknown source function [either $\tilde{I}(\mathbf{k})$ or $\tilde{g}(\mathbf{K})$] can be readily determined from far field interference experiments. To see this, let us consider a Young's interference experiment[14] performed on the surface of a sphere of radius R as illustrated in Fig. 1. The interference fringes observed in the vicinity of the central point P on the observation screen allows one to determine the spatial coherence function of the field at the points P_1,P_2 in the aperture plane. In the wave zone ($k_0R\to\infty$) this coherence function is proportional to the autocorrelation function of the radiation pattern evaluated in the directions $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ so that from this experiment we determine[15]

$$\tilde{I}[k_0(\hat{\mathbf{r}}_2-\hat{\mathbf{r}}_1)]\tilde{g}\left[k_0\left(\frac{\hat{\mathbf{r}}_1+\hat{\mathbf{r}}_2}{2}\right)\right]=\langle\hat{\psi}^*(\hat{\mathbf{r}}_1)\hat{\psi}(\hat{\mathbf{r}}_2)\rangle. \quad (4.6)$$

By varying the locations of the pinholes at points P_1,P_2 the wave vector $\mathbf{k}=k_0(\hat{\mathbf{r}}_2-\hat{\mathbf{r}}_1)$ can be made to assume all values within a sphere of radius $2k_0$ while the wave vector

$\mathbf{K} = k_0(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2)/2$ can be made to assume all values within a sphere of radius k_0. It follows that at the very least a band-limited approximation to one of the functions can be determined if the other is known. When applied to the problem of determining the intensity profile of a source of known coherence, the procedure is quite analogous to that used in determining the intensity profile of stellar sources by means of the Michelson stellar interferometer.[16] This latter procedure requires, however, that the stellar source be approximated by a *planar*, *incoherent* source so that the Van Cittert–Zernike theorem can be applied,[17] whereas in the procedure outlined above it is only necessary to assume that the stellar source can be approximated by a quasihomogeneous source of known coherence properties.

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[3]An account of the early work in the theory of nonradiating sources is included in D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948). More recent developments are reported by T. Erber and S.M. Prastein, Acta Phys. Austr. **32**, 224 (1970); J.B. Arnett and G.H. Goedecke, Phys. Rev. **168**, 1424 (1968); A.J. Devaney and E. Wolf, Phys. Rev. D **8**, 1044 (1973) and B.J. Hoenders, *Inverse Source Problems in Optics*, edited by H.P. Baltes (Springer-Verlag, Heidelberg, 1978) pp. 41–82. The importance of nonradiating sources to uniqueness in the deterministic inverse source problem is discussed by C. Müller, IRE Trans. Antennas Propag. **AP-4**, 3, 224 (1956) and by H. Bleistein and J.K. Cohen, J. Math. Phys. **18**, 194 (1977). The relevance of these sources to the inverse scattering problem is demonstrated in A.J. Devaney, J. Math. Phys. **19**, 1526 (1978).

[4]Throughout the paper ensemble averages will be denoted by angular brackets $\langle\ \rangle$ enclosing the quantity to be averaged and complex conjugates by a superscript asterisk * on the quantity being conjugated.

[5]In some applications the field $\psi(\mathbf{r})$ is the Fourier amplitude at frequency $\omega_0 = ck_0$ of a stationary time dependent field. For such cases the definition of the autocorrelation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ given in Eq. (1.5) has to be modified to include an integration over an infinitesmal band of frequencies surrounding ω_0 and the resulting quantity is referred to as the cross-spectral density function of the *time dependent* field. [See, for example, the discussion presented in E.W. Marchand and E. Wolf, J. Opt. Soc. Am. **62**, 379 (1972).]

[6]The quantity $\Phi(\mathbf{k}_1, \mathbf{k}_2)$ is, of course, simply the sixfold Fourier transform of the source autocorrelation function $R_p(\mathbf{r}_1, \mathbf{r}_2)$ and is, thus, the *generalized spectral density function* of the source random process. [See, for example, R.A. Silverman, IRE Trans. Inf. Theory **3**, 182 (1957).]

[7]Quasihomogeneous two-dimensional sources were introduced by W.H. Carter and E. Wolf [J. Opt. Soc. Am. **67**, 785 (1977)] and find many applications in the modern theory of radiometry [cf. E. Wolf and E. Collett, Opt. Comm. **25**, 293 (1978); E. Wolf, J. Opt. Soc, Am. **68**, 6 (1978)]. Three-dimensional quasihomogeneous sources and the properties of the fields they generate will be discussed in a forthcoming paper by E. Wolf and W.H. Carter.

[8]P.M. Morse and K.U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), Chap. 7.

[9]The analyticity of $\tilde{\rho}(\mathbf{k})$ follows from a three-dimensional version of a well-known theorem that the Fourier transform of a continuous function which vanishes outside a finite interval is an entire analytic function. The multidimensional form of the theorem is the Plancherel-Polya theorem (cf. Ref. 10, p. 352).

[10]B.A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables* (American Mathematics Society, Providence, Rhode Island, 1963).

[11]An analogous situation occurs in the inverse scattering problem where it is possible to have *nonscattering* potentials within the scattering volume (cf. the paper by this author quoted in Ref. 3).

[12]An account of the theory of random nonradiating sources will be presented in a forthcoming paper by B.J. Hoenders and H.P. Baltes.

[13]The procedure used in converting from the $(\mathbf{r}_1, \mathbf{r}_2)$ variables to the $(\mathbf{R}, \mathbf{r})$ variables and the introduction of the frequency variables $(\mathbf{k}, \mathbf{K})$ defined in Eq. (4.4) is entirely analogous to the treatment presented in Reference 7 for two-dimensional sources.

[14]M. Born and E. Wolf, *Principles of Optics*, 3rd ed. (Pergamon, New York, 1965), Sec. 7.2.

[15]The relationship (4.6) between the transforms $\tilde{I}$ and $\tilde{g}$ and the autocorrelation function of the radiation pattern is a three-dimensional version of the generalized Van Cittert–Zernike theorem (A. Walther, J. Opt. Soc. Am. **58**, 1256 (1968); E.W. Marchand and E. Wolf, J. Opt. Soc. Am. **62**, 379 (1972) as applied to quasihomogeneous planar sources (cf., the paper by Carter and Wolf quoted in Ref. 7). See also: H.P. Baltes and B. Steinle, Lett. Nuovo Cimento **18**, 313 (1977).

[16]See, for example, L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

[17]The requirement that the stellar source be incoherent can be dropped if the generalized Van Cittert–Zernike theorem (see Ref. 15) is used rather than the classical form of this theorem (cf. Ref. 14, Sec. 10.4.2).