Diffraction Tomography I: The Fourier Diffraction Theorem

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Diffraction tomography I

- The Fourier diffraction theorem: Green’s function decomposition
- The Fourier diffraction theorem: The Fourier transform approach
- A limit of the Fourier diffraction theorem
- The Fourier space coverage discussion (synthetic aperture)

We will discuss interpolation methods and filtered backpropagation methods in the next lecture.
What we are trying to understand...

**Diffraction tomography vs. Projection tomography**

- **Detector plane** $u_p(x)$
- **Object o(x,y)**
- **Undiffracted field** (Fourier projection slice theorem)
- **Illumination**

- **Detector plane** $u_s(x)$
- **Object o(x,y)**
- **Diffracted field**
- **Plane wave**
Tomography with diffracted or scattered fields

Relationship between the object $o(r)$ and diffracted field: $(r = (x, y))$

- X-ray projection ($u_p$): undiffracted field (projection)
  \[ u_p(x) = \int \chi o(r) dy \]

- EM, acoustic ($u_d$): diffracted field
  \[ u(r) = u_0(r) + u_d(r) \]
  \[ (\nabla^2 + k_0^2)u_d(r) = -o(r)u(r): \text{scalar Helmholtz equation} \]
  \[ o(r) = k_0^2[n^2(r) - 1]: \text{object scattering density} \]
  \[ u_d(r) = \int g(r|r') o(r') u(r') dr', \]
  \[ g(r|r') = \frac{\exp(jk_0|r - r'|)}{4\pi|r - r'|}: \text{green's function} \]
The first Born approximation

\[ u(r) = u_0(r) + u_d(r) \]

Assumption: \( u_d \ll u_0 \): weakly scattering object

**The first Born approximation**

\[
u_d(x) = \int g(r - r')o(r')u_0(r')dr' + \int g(r - r')o(r')u_d(r')dr' \approx \int g(r - r')o(r')u_0(r')dr'
\]
The Fourier diffraction theorem

$u_0$: incident plane wave
$u_d$: diffracted (or scattered) field

**Theorem:** When an object $o$ is illuminated by a plane wave $u_0$, the Fourier transform of the diffracted field produces the Fourier transform $O$ of the object along a semicircular arc in 2-D and along the semispherical surface in 3-D in the spatial frequency domain.

\[ F\{u_d(y)\} \]
The Fourier diffraction theorem proof: plane wave decomposition of Green’s function

Plane wave decomposition of Green’s function

\[ g(r|r') = g(r - r') = \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \frac{1}{\beta} \exp \{ j[\alpha(x - x') + \beta|y - y'|] \} \]

\[ u_d(r) = \int dr' o(r) u_0(r) \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \frac{1}{\beta} \exp \{ j[\alpha(x - x') + \beta|y - y'|] \} \]

\[ u_0(r) = \exp(js_0 \cdot r) \]

\[ s_0 = (0, k_0) \]

\[ l_0 > y' \]
\alpha^2 + \beta^2 = k_0^2

\[u_d(x, y = l_0) = \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \int d\mathbf{r}' \frac{o(\mathbf{r}')}{\beta} \exp \left\{ j[\alpha(x - x') + \beta(l_0 - y')] \right\} \exp \left\{ jk_0y' \right\}\]

\[= \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \int d\mathbf{r}' \frac{o(\mathbf{r}')}{\beta} \exp \left\{ -j[\alpha x' + (\beta - k_0)y'] \right\} \exp \left\{ -j(\alpha x + \beta l_0) \right\}\]

\[= \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \frac{\exp \left\{ -j(\alpha x + \beta l_0) \right\}}{\beta} \int d\mathbf{r}' o(\mathbf{r}') \exp \left\{ -j[\alpha x' + (\beta - k_0)y'] \right\}\]

\[= \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \frac{\exp \left\{ -j(\alpha x + \beta l_0) \right\}}{\beta} O(\alpha, \beta - k_0)\]
\[ \alpha^2 + \beta^2 = k_0^2 \]

\[ F\{u_d(x, y = l_0)\} = \int_{-\infty}^{\infty} dx u_d(x, y = l_0) \exp(-j\alpha x) \]

\[ = \frac{j}{4\pi} \int_{-\infty}^{\infty} dx \int d\alpha' \frac{\exp\{-j(\alpha' x + \beta l_0)\}}{\beta} O(\alpha', \beta - k_0) \exp(-j\alpha x) \]

\[ = \frac{j}{4\pi} \int_{-\infty}^{\infty} dx \int d\alpha' \exp\{j(\alpha' - \alpha)x\} \frac{\exp(j\beta l_0)}{\beta} O(\alpha', \beta - k_0) \]

\[ = \frac{j}{4\pi} \int d\alpha' 2\pi \delta(\alpha' - \alpha) \frac{\exp(j\beta l_0)}{\beta} O(\alpha', \beta - k_0) \]

\[ = \frac{j}{2} \frac{\exp(j\beta l_0)}{\beta} O(\alpha, \beta - k_0) = \frac{j}{2} \frac{\exp(jl_0 \sqrt{k_0^2 - \alpha^2})}{\sqrt{k_0^2 - \alpha^2}} O(\alpha, \sqrt{k_0^2 - \alpha^2} - k_0) \]
The Fourier diffraction theorem proof: plane wave decomposition of Green’s function (Cont’d)

\[ u_0(\mathbf{r}) = \exp(j \mathbf{s}_0 \cdot \mathbf{r}) \]
\[ \mathbf{s}_0 = (k_0 \cos \theta, k_0 \sin \theta) \]
\[ l_0 > y' \]

\[ u_d(x, y = l_0) \]
\[ = \frac{j}{4\pi} \int_{-\infty}^{\infty} d\alpha \exp \left\{ -j(\alpha x + \beta l_0) \right\} \frac{O(\alpha - k_0 \cos \theta, \beta - k_0 \sin \theta)}{\beta} \]

\[ U_d(\alpha) = \mathcal{F} \{ u_d(x, y = l_0) \} \]
\[ = \frac{j \exp(j l_0 \sqrt{k_0^2 - \alpha^2})}{2 \sqrt{k_0^2 - \alpha^2}} O(\alpha - k_0 \cos \theta, \sqrt{k_0^2 - \alpha^2} - k_0 \sin \theta) \]
The Fourier diffraction theorem proof: plane wave decomposition of Green’s function (Cont’d)
The Fourier diffraction theorem: The Fourier transform approach

\[ u_d(x) = \int g(r - r') o(r') u_0(r') dr' \]

\[ U_d(\alpha, \beta) = G(\alpha, \beta) \left[ F\{ o(r') u_0(r') \} \right] \]
\[ = G(\alpha, \beta) \left[ O(\alpha, \beta) \ast U_0(\alpha, \beta) \right] \]

- \( G(\alpha, \beta) \): Fourier transform of the scalar Helmholtz equation

\[ (\nabla^2 + k_0^2) g(r|r') = -\delta(r - r') \]
\[ (-\omega^2 + k_0^2) G(\omega|r') = -\exp(-j\omega \cdot r') \]
\[ \omega = (\alpha, \beta), \ k_0^2 = \alpha^2 + \beta^2 \]
The Fourier diffraction theorem: The Fourier transform approach (Cont’d)

- $U_0(\alpha, \beta)$: Fourier transform of the incident plane wave

$$U_0(\omega) = U_0(\alpha, \beta) = \mathcal{F}\{\exp(js_0 \cdot r')\} = 2\pi \delta(\omega - s_0)$$

$$O(\alpha, \beta) \ast U_0(\alpha, \beta) = 2\pi O(\alpha - k_0 \cos \theta, \beta - k_0 \sin \theta)$$

$$G(\alpha, \beta | r' = 0) = \frac{-1}{k_0^2 - \alpha^2 - \beta^2}$$

Green’s function Fourier transform has two poles at $k_0 = \pm \sqrt{\alpha^2 + \beta^2}$.

$$U_d(\alpha, \beta) = \frac{2\pi O(\alpha - k_0 \cos \theta, \beta - k_0 \sin \theta)}{\alpha^2 + \beta^2 - k_0^2}$$
The Fourier diffraction theorem: The Fourier transform approach (Cont’d)

Diffracted field along the detector:

\[ u_d(x, y = l_0) = \frac{1}{4\pi^2} \iint d\alpha d\beta U_d(\alpha, \beta) \exp\{j(\alpha x + \beta l_0)\} \]

\[ = \frac{1}{4\pi^2} \iint d\alpha d\beta \frac{O(\alpha - k_0 \cos \theta, \beta - k_0 \sin \theta)}{\alpha^2 + \beta^2 - k_0^2} \exp\{j(\alpha x + \beta l_0)\} \]

Two poles:

\[ \beta = \pm \sqrt{k_0^2 - \alpha^2} \]

Contour integration with residues at the two poles:

\[ u_d(x, l_0) = \frac{1}{2\pi} \int d\alpha \Gamma_1(\alpha; l_0) \exp(j\alpha x) + \frac{1}{2\pi} \int d\alpha \Gamma_2(\alpha; l_0) \exp(j\alpha x) \]
The Fourier diffraction theorem: The Fourier transform approach (Cont’d)

\[
\Gamma_1(\alpha, l_0) = \frac{jO(\alpha - k_0 \cos \theta, \sqrt{k_0^2 - \alpha^2 - k_0 \sin \theta})}{2 \sqrt{k_0^2 - \alpha^2}} \exp(jl_0\sqrt{k_0^2 - \alpha^2})
\]

\[
\Gamma_2(\alpha, l_0) = \frac{-jO(\alpha - k_0 \cos \theta, -\sqrt{k_0^2 - \alpha^2 - k_0 \sin \theta})}{2 \sqrt{k_0^2 - \alpha^2}} \exp(-jl_0\sqrt{k_0^2 - \alpha^2})
\]
The Fourier diffraction theorem: The Fourier transform approach (Cont’d)

\[ u_d(x, l_0) = \frac{1}{2\pi} \int d\alpha \Gamma_1(\alpha; l_0) \exp(j\alpha x) \quad \text{transmission geometry} \]

\[ u_d(x, l_0) = \frac{1}{2\pi} \int d\alpha \Gamma_2(\alpha; l_0) \exp(j\alpha x) \quad \text{reflection geometry} \]
\( \lambda_0 \rightarrow 0 \), equivalently, \( k_0 = \frac{2\pi}{\lambda_0} \rightarrow \infty \)

The radius of the \( k \)-circle is determined by the wavelength \( \lambda_0 \).

\[ k_2 > k_1 \text{ (X-ray vs. visible ray)} \]

- X-ray: radius = \( 5 \times 10^8 \) rads/meter
  - resolution < 1000 rads/meter
Data collection: synthetic aperture discussion

Frequency coverage discussion

- Multispectral illumination
- Multiangle illumination
- Vertical seismic profiling (VSP): Figs. 6.11 and 6.12
- Synthetic aperture approach: Figs. 6.9 and 6.10
Filter and point spread function

\[
\text{ring}(\alpha, \beta) \longleftrightarrow \pi J_0(\pi \sqrt{\alpha^2 + \beta^2})
\]

\[
\text{ring}(\alpha, \beta) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{rect} \left[ \frac{2 \sqrt{\alpha^2 + \beta^2} - 1}{2\epsilon} \right]
\]

\[
H(\alpha, \beta) = \text{ring}(\alpha, \beta) \text{rect} \left( \frac{\alpha}{a}, \frac{\beta}{b} \right)
\]

\[
h(x, y) = \pi J_0(\pi \sqrt{x^2 + y^2}) * \text{sinc}(ax, by)
\]
Point spread function
An estimate of resolution from a linear filtering perspective

\[
\begin{align*}
\Delta_x &= \frac{2\pi}{A} = \frac{\lambda}{2NA} \\
\Delta_y &= \frac{2\pi}{B} = \frac{2\lambda}{NA^2}
\end{align*}
\]
Simulations: transmission vs. reflection
Simulations: transmission vs. reflection